Vygotskian scientific concepts and connectionist teaching in mathematics
David Swanson

Abstract:
This paper explores Vygotsky’s theory of scientific concepts. It does so in direct relation to forms of mathematical pedagogy which strive to develop more meaningful activity than the dominant drill and memory-based practice. In doing so it develops the beginnings of a systematic theoretical explanation, and justification, of such approaches to teaching.

Keywords: Vygotsky, scientific concepts, mathematics pedagogy, connectionism

1 Introduction
This article is an exploration of Vygotsky’s theory of scientific concepts, motivated by its particular relevance to age-old debates about mathematical pedagogy. The much maligned, yet dominant in practice, rote-learning devoid of meaning which has been exacerbated in recent times by an increased emphasis on exam results and atomisation of the curriculum (e.g. see Gainsburg, 2012), was clearly prevalent in Vygotsky’s (1994) time too:

Educational experience, no less than theoretical research, teaches us that, in practice, a straightforward learning of concepts always proves impossible and educationally fruitless. Usually, any teacher setting out on this road achieves nothing except a meaningless acquisition of words, mere verbalization in children, which is nothing more than simulation and imitation of corresponding concepts which, in reality, are concealing a vacuum. In such cases, the child assimilates not concepts but words, and he fills his memory more than his thinking. As a result, he ends up helpless in the face of any sensible attempt to apply any of this acquired knowledge. Essentially, this method of teaching/learning concepts, a purely scholastic and verbal method of teaching, which is condemned by everybody and which advocates the replacement of acquisition of living knowledge by the assimilation of dead and empty verbal schemes, represents the most basic failing in the field of education (p. 356).

But more than sharing an awareness of the problem, Vygotsky also developed a theory containing key elements which map across to the main aspects of pedagogical approaches
which oppose the dominant transmissionism and encourage more meaningful activity in the mathematics classroom. There have, of course, been many such approaches, at different times and in different places, in both research and practice. One such approach, connectionist teaching (Askew, Brown, Rhodes, William, & Johnson, 1997), exemplifies many of the elements found in these and provides a terminology which fits aptly with the theory to be discussed below. The central aspects of connectionist teaching include an emphasis on:

- Meaningful problems,
- Connecting mathematics to the real world,
- The connections within mathematics,
- Connecting to students’ existing knowledge,
- Dialogue, reasoning and justification,
- Reflexivity.

The central strands of Vygotsky’s theory of scientific concepts, on the other hand, are the following:

- The relationship of concepts to activity and real problems,
- The relationship between, and merging of, abstract and concrete in concepts,
- The importance of the scientific concept’s place within a definite system,
- The relationship between everyday and scientific concepts,
- The key role in concept formation played by the word and dialogue,
- The relationship of conscious awareness and control to generalisation within systems.

An initial comparison of these two lists may already provoke an awareness of the similarities between them. The aim of what follows is to draw out and deepen these connections and show how Vygotsky’s theory is well placed to provide an effective theoretical justification for, and explanation of the relative success of, connectionist teaching methods.

2 Scientific concepts

The term ‘scientific’ can often carry implications of rationalism, empiricism or, perhaps, a narrow restriction to the physical sciences. This was not Vygotsky’s intention, as is partly indicated by his inclusion of terms such as ‘exploitation’ and ‘class struggle’ under the designation. What Vygotsky does mean by ‘scientific’ however will take time to unfold through what follows. To assist in that process, substantial use will be made of Vygotsky’s own writings.

There are two key works by Vygotsky which are available in English on the subject of concepts. The first is the chapter on ‘The development of thinking and formation of concepts in the adolescent’ from ‘Pedology of the adolescent’ (1998), written around 1930-31 (p. 319). A section of this appears as Chapter 5 in ‘Thinking and Speech’ (1987). The second is Chapter 6 of Thinking and Speech (1987), written around 1933-34 (p. 376) on ‘The development of scientific concepts in children’. This second work extends the arguments of the earlier writing and introduces the distinction between everyday and scientific concepts.

Although not explicitly focused on the psychology of concept development, there are other works by Vygotsky which are useful in developing a more complete and historical view of the
theory. For example, an earlier work, ‘The historical meaning of the crisis in psychology’ (1997) written around 1926-27 (p. 14) raises some themes in relation to the development of a general science of psychology which re-emerge in the discussion of scientific concepts. Also, in ‘The history of the development of higher mental functions’ (1997), written approximately in 1931 (p. 279), the wider psychological system of which thinking in concepts is part is discussed more fully. In this, these higher aspects, such as voluntary attention, logical memory and volition, are seen to join with the development of concepts as part of a complex interrelating whole, the development of which is stimulated by the social and cultural with a key role played in the process by signification and, in particular, the word.

For Vygotsky, thinking in concepts represents ‘a new and higher form of thinking’ which appears in adolescence (1998, p.38) due to the influence of external culture, and in particular, schooling:

[W]hen educational material is assimilated that consists for the most part of general positions that express some law or rule, through the influence of speech, attention is diverted more and more in the direction of abstract relations and thus leads to the formation of abstract concepts.

This new content ‘of necessity requires [the adolescent’s] transition to new forms, and places before her’ problems that can be resolved only through the formation of concepts’. This emphasis on social factors in the integrated development of thought, attention, memory and will prevents the theory from becoming ‘cognitivist’ despite its clear focus on cognition.

The importance of the social world posing problems in leading development is an approach shared by connectionist pedagogy, and this is a useful starting point for exploring the links between connectionism and Vygotsky’s theory.

2.1 Meaningful problems

[T]he right perspective is primarily from environment towards mathematics rather than the other way round. Not: first mathematics and then back to the real world but first the real world and then mathematising.

The real world - what does it mean? Forgive this careless expression. In teaching mathematising ‘the real world’ is represented by a meaningful context involving a mathematical problem. 'Meaningful' of course means: meaningful to the learners. Mathematics should be taught within contexts, and I would like the most abstract mathematics taught within the most concrete context (Freudenthal 1981, p. 144).

The importance of using stimulating problems, whether real in the literal sense of real-world or real in the sense of meaningful and genuine is hopefully obvious in terms of helping to engage and motivate learners. It is also essential for concept development, in creating a need for the concept:

The concept does not live in isolation… it is not a congealed, static formation but a formation that is always encountered in the vital and complex process of thinking. A

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1 The translation of general pronouns in Vygotsky’s work has been occasionally altered for gender balance.
concept always fulfils some function in communication, reasoning, understanding, or problem solving (Vygotsky, 1987, p. 123).

This functional role of the concept means that it cannot appear simply through associative connection:

[T]he concept arises and is formed in a complex operation that is directed toward the resolution of some task… In itself, learning words and their connections with objects does not lead to the formation of concepts. The subject must be faced with a task that can only be resolved through the formation of concepts (1987, p. 124).

For Vygotsky, this is a necessary but not sufficient condition for the appearance of concepts in development, with the changing functional role of the word in adolescence instead being the key factor. For the development of particular concepts in mathematics for an individual, the necessity still holds though. As frequently illustrated within the situated cognition (Lave 1993) literature, humans can, and frequently do, find all sorts of associative connections to more simply and effectively achieve their aims (see, for example, Hoyles, Noss & Pozzi, 2001, on how nurses deal with proportion and medicine in practice). To go beyond this, amongst other things, requires a good reason or a failure of those associative methods to work. It can be argued that the transmissionist approach in education with its repetitive practice of short, meaningless questions encourages just such a situated associative approach. Greiffenhagen & Sharrock (2008) for example, have shown the similarity of school and situated mathematics (however, without the argument here that there is something that lies beyond both). The use of meaningful problems on the other hand, is more likely to stimulate the need for, as well as providing the motivation for, concept development.

2.2 The abstract-concrete relationship

Attempting to connect formal mathematics to the real world within pedagogy has several interrelated aims- to make the mathematics more meaningful, to aid motivation and to increase understanding (see, for example, Gainsburg, 2008). The ‘real world’ appearing in mathematical problems can mean a wide range of things however, even if illusory connections in questions such as, ‘This table top is in the shape of a trapezium, what is its area?’ are ignored. It could include Freirean links to understanding social inequality, or the relation of mathematics to other subjects of interest to students’ everyday lives. It can also include modelling or grounding mathematics directly in social, physical and visual experience (as in, for example, children experiencing the concept of circle as the loci of fair throwing points in a game, see Doig, Groves & Fujii, 2011). This section will focus primarily on the latter type of connections. However, what could be termed the less immediate connections will feature in later sections.

In Vygotsky’s theory, the relationship between the abstract and the concrete is central to understanding the nature of concepts. By ‘concrete’, Vygotsky essentially refers to physical, spatio-temporal and visual perception and direct experience of social activity:

[F]or the child in particular, the concept is linked with sensual material, the perception and transformation of which gives rise to the concept itself. This sensual material and the word are both necessary for the concept’s development (1987, p. 111)
The emphasis Vygotsky places on the relationship between abstract and concrete in concept development is pre-figured in his earlier writings on the science of psychology:

Every natural-scientific concept, however high the degree of its abstraction from the empirical fact, always contains a clot, a sediment of the concrete, real and scientifically known reality, albeit in a very weak solution, i.e., to every ultimate concept, even to the most abstract, corresponds some aspect of reality which the concept represents in an abstract, isolated form. Even purely fictitious, not natural-scientific but mathematical concepts ultimately contain some echo, some reflection of the real relations between things and the real processes, although they did not develop from empirical, actual knowledge, but purely a priori, via the deductive path of speculative logical operations. As Engels demonstrated, even such an abstract concept as the series of numbers, or even such an obvious fiction as zero, i.e., the idea of the absence of any magnitude, is full of properties that are qualitative, i.e., in the end they correspond in a very remote and dissolved form to real, actual relations. Reality exists even in the imaginary abstractions of mathematics (1997, p.248).

At the same time,

[E]ven the most immediate, empirical, raw, singular natural scientific fact already contains a first abstraction. The real and the scientific fact are distinct in that the scientific fact is a real fact included into a certain system of knowledge, i.e., an abstraction of several features from the inexhaustible sum of features of the natural fact. The material of science is not raw, but logically elaborated, natural material which has been selected according to a certain feature. Physical body, movement, matter – these are all abstractions. The fact itself of naming a fact by a word means to frame this fact in a concept, to single out one of its aspects; it is an act toward understanding this fact by including it into a category of phenomena which have been empirically studied before... Everything described as a fact is already a theory (p. 249).

This emphasis on the role of the system in which a concept belongs appears more strongly in what can be called the second phase of Vygotsky's work on concepts, where a differentiation occurs between everyday and scientific concepts. This will be discussed more fully in the sections which follow. In the first stage, experimental analysis leads Vygotsky (1987) to a categorisation of pre-conceptual forms in development which are distinguished by the nature of the generalisation involved. This includes syncretic generalisation which is based on subjective factors, various forms of complexes which rely on objective connections in perception and activity, and pseudo-concepts which are externally similar to concepts but still based on concrete connections. ‘True’ concepts are distinguished by generalisation on the basis of abstraction.

The unique intellectual formations present in the pre-adolescent period are, in fact, functionally equivalent to the true concepts that mature later. They fulfil a function similar to that of concepts and function in the resolution of similar tasks. However, experimental analysis indicates that their psychological nature, their constituents, their structure, and their mode of activity differ significantly from those of the true concept (p. 130).
In tracing these earlier forms the merging roles of generalisation and abstraction are uncovered:

The concept ... develop[s] along two different channels. First... the function of combining or connecting a series of separate objects through a common family name is basic to the child’s complexive thinking. This constitutes the first of the two channels... [P]otential concepts, concepts which are based on the isolation of several common features, develop in parallel with complexes and constitute the second channel. These two forms constitute the dual roots of concept formation. (1987, p. 165)

What is important about the 'abstract' in developed conceptual thought, formed through generalisation on the basis of abstraction is that this abstraction should not be seen simply as the shedding of non-essential features:

the claim that the abstract thinking of the adolescent breaks away from the concrete, and the abstract from the visual is incorrect: the movement of thinking during this period is characterized not by the intellect's breaking the connections to the concrete base which it is outgrowing, but by the fact that a completely new form of relation between abstract and concrete factors in thinking arises, a new form of their merging and synthesis, that at this time elementary functions long since established, such as visual thinking, perception, or the practical intellect of the child appear before us in a completely new form (1998, p.37).

The concrete connections still remain part of the abstract but a new relationship based on the inner connections between things is also fused with it. In a sense this brings thought closer to reality and Vygotsky quotes Lenin making a similar point:

In opposition to Kant, Hegel was essentially completely correct. Thinking going from the concrete to the abstract does not deviate – if it is correct... from truth, but approaches it. The abstraction of material, a law of nature, abstraction of value, etc., in a word, all scientific (correct, serious, not foolish) abstractions reflect nature more deeply, more reliably, more fully. From a living contemplation to abstract thinking and from it to practice – such is the dialectical path of recognizing truth, recognizing objective reality (p. 79).

In this understanding, the development of abstract concepts necessarily requires the existence of rich, concrete experience from which to abstract and generalise from. Pedagogies which attempt to transmit an understanding of formal mathematics without this inevitably suffer:

It is completely clear that if the process of generalizing is considered as a direct result of abstraction of traits, then we will inevitably come to the conclusion that thinking in concepts is removed from reality. ... Others have said that concepts arise in the process of castrating reality. Concrete, diverse phenomena must lose their traits one after the other in order that a concept might be formed. Actually what arises is a dry and empty abstraction in which the diverse, full-blooded reality is impoverished by logical thought. This is the source of the celebrated words of Goethe: 'Gray is every theory and eternally green is the golden tree of life' (1998, p53).
Instead:

A real concept is an image of an objective thing in its full complexity. Only when we recognize the thing in all its connections and relations, only when this diversity is synthesized in a word, in an integral image through a multitude of determinations, do we develop a concept. According to the teaching of dialectical logic, a concept includes not only the general, but also the individual and the particular. In contrast to contemplation, to direct knowledge of an object, a concept is filled with definitions of the object; it is the result of a rational processing of our experience, and it is a mediated knowledge of the object. To think of some object with the help of a concept means to include the given object in a complex system of mediating connections and relations disclosed in determinations of the concept. Thus the concept does not arise from this as a mechanical result of abstraction – it is the result of a long and deep knowledge of the object (1998, p.53).

One counter argument to enriching mathematical activity with strong concrete context is the ability of a minority of students to succeed in its absence, or, the activity of the many professional mathematicians who get by just fine without any thought of the real world (see, e.g. Burton, 1998, for examples of such mathematicians discussing their activity). This is perhaps because devoting large amounts of time to mathematics is enough to make it as richly experiential and 'concrete' as the real world is for the majority. Most though are alienated long before this happens.

A second more complex counter argument is that an overemphasis on the concrete can inhibit the need for developing specifically mathematical or logical forms of thought (see for example, Davydov, 1990, p.60). Again, the situated cognition literature reinforces this point with its many examples of the success of thinking which is tied to the context. The use of problems which require new concepts to solve are one feature which can prompt going beyond this which has already been discussed. Another though is an emphasis on the system to which concepts belong.

2.3 Connections in a system

In a recent research project involving the author, one secondary school student, when asked which area of mathematics she enjoyed the most, responded, 'I don't mind doing brackets'. Aside from the admirable unwillingness to be too enthusiastic, the notion that 'brackets' are an area of mathematics offers an illustrative reflection of the atomised nature of the curriculum. Meaningful, open-ended problems are one method of overcoming this 'bite-size' approach, for example, in giving the opportunity to compare alternative methods. Simply making the connections between various aspects of the curriculum would in itself be a helpful starting point though. To give one example, a large part of the GCSE mathematics curriculum in the UK involves proportional thinking. This includes equivalent fractions, pie charts, similar triangles, 'best buys', trigonometry and other traditional topics. Yet the potential connections between these are missed in a fragmented approach to the curriculum.

Vygotsky's theory points to the importance of these systemic connections in concept development. Scientific concepts are seen not only to merge abstract and concrete, as described in the previous section, but also to include the relationships between concepts:
The concept does not emerge in the child’s mind like a pea in a sack. Concepts do not lie alongside one another or on top of one another with no connections or relationships. If this were the case, thought operations requiring the co-relation of concepts would be impossible, as would the child’s world view and the entire complex life of his thought. Moreover, without well-defined relationships to other concepts, the concept’s existence would be impossible. In contrast to what is taught by formal logic, the essence of the concept or generalization lies not in the impoverishment but in the enrichment of the reality that it represents, in the enrichment of what is given in immediate sensual perception and contemplation. However, this enrichment of the immediate perception of reality by generalization can only occur if complex connections, dependencies, and relationships are established between the objects that are represented in concepts and the rest of reality. By its very nature, each concept presupposes the presence of a certain system of concepts. Outside such a system, it cannot exist (1987, p. 224).

One simple example which illustrates this:

The group of nine on playing cards is richer and more concrete than our concept of ‘9’, but the concept ‘9’ involves a number of judgements which are not in the nine on the playing card; ‘9’ is not divisible by even numbers, is divisible by 3, is $3^2$, and the square root of 81; we connect ‘9’ with the series of whole numbers, etc. Hence it is clear that psychologically speaking the process of concept formation resides in the discovery of the connections of the given object with a number of others, in finding the real whole. That is why a mature concept involves the whole totality of its relations, its place in the world, so to speak. ‘9’ is a specific point in the whole theory of numbers with the possibility of infinite development and infinite combination which are always subject to a general law...We see that the concept is a system of judgements brought into a certain lawful connection: the whole essence is that when we operate with each separate concept, we are operating with the system as a whole (1997, p.100).

At the heart of systematisation is again the act of generalising. To give a simplistic example, if the word 'dog' represents the generalisation through abstraction of the things that are dogs, the word 'animal' in turn generalises this generalisation.

A new stage in the development of generalization is achieved only through the reformation - not the nullification – of the previous stage. The new stage is achieved through the generalization of the system of objects already generalized in the previous stage, not through a new generalization of isolated objects. The transition from preconcepts (e.g., the school child’s arithmetic concept) to true concepts (e.g., the adolescent’s algebraic concept) occurs through the generalization of previously generalized objects (1987, p. 230).

Building this intricate, interrelated conceptual system is a key task in mathematics education. In the earlier example, making connections between the various proportion-related topics in GCSE mathematics would represent another (relatively simple) example of generalising generalisations and developing the conceptual system. Such systematising generalisations are seen by Vygotsky as allowing more freedom of movement in thought:
[T]he growth of algebraic generalizations is accompanied by a growth in the freedom of operations. The process involved in the liberation from links with the numerical field occurs differently than the process involved in the liberation from links with the visual field. The growth in freedom that occurs with the emergence of the algebraic generalization is explained by the potential for reverse movement from the higher stage to the lower that is inherent in the higher generalization; the lower operation is already viewed as a special case of the higher. Arithmetic operations are preserved even after algebra is learned. This naturally leads to the question of what differentiates the arithmetic concept of the adolescent who has mastered algebra from that of the school child who has not. Research indicates that the adolescent views the arithmetic concept as a special case of the more general algebraic concept. Research also indicates that operations with the arithmetic concept become freer. Because of its independence from particular arithmetic expressions, it is applied in accordance with a more general formula (1987, p. 230).

In later sections this positive effect of systematisation is explored further, first, in terms of how the higher generalisations reach down and reshape what is generalised, and, second, in Vygotsky’s equating of generalisation with conscious awareness (and therefore conscious control). Before moving on however, it is worth reflecting on some points related to the following:

In school, the child does not learn the decimal system as such. He learns to write numbers, add, multiply, and solve problems. Nonetheless, some general concept of the decimal system does develop (1987, p. 207).

For analytical purposes, this article as a whole paints a picture of two extremes in pedagogy, connectionism and transmissionism. In practice, many of the aspects attributed to connectionism appear in its portrayed opposite in diluted form. So, for example, in terms of systematisation even the worst textbook will structure the material in some kind of logical way, building on what has come before, and connected topics will be met temporally close to one another even if the links between them are not made explicit. Given the ‘real problem’ of making sense of the material, some students will generalise to some extent. So the real argument here is not that this sort of thing is impossible under those conditions but rather that it is more worthwhile to assist this process than to put obstacles in its way. This two-way process, in a sense from-above and from-below, is central to Vygotsky’s understanding of the role of the social in leading development in a form that isn't purely about one-way internalisation. Nevertheless, the role of the social remains fundamental and in concept formation this role is primarily mediated by words.

2.4 Word, Dialogue, Justification

To discuss concepts is to discuss words. ‘Psychologically, the development of concepts and the development of word meaning are one and the same process ‘(1987, p. 180). In development, it is the changing role of the word which, for Vygotsky, is the key factor:

[A]ssociation, attention, representation, judgement, goal are all indispensable functions in concept development but all exist prior to adolescence. ‘what is central
to this process is the functional use of the sign or word as the means through which the adolescent masters and subordinates his own mental operations and directs their activity in the resolution of the tasks that face him (1987, p. 131).

Words do this through acting as a 'means of actively directing attention, partitioning and isolating attributes, abstracting these attributes, and synthesizing them'. In pre-conceptual forms,

[T]he child’s complexes (which correspond to word meanings) do not develop freely or spontaneously along lines demarcated by the child herself. Rather, they develop along lines that are preordained by the word meanings that have been established in adult speech (1987, p. 142).

And it is in this shared use of the word and,

[I]n the associated abstraction of distinct features, that the child first destroys the concrete situation and the concrete connections among the object’s features. In this process, she creates the prerequisites for the unification of these features on a new foundation (1987, p. 159).

Social dialogue continues to play an ongoing role in conceptual development through this function of the word but also in its wider relationship to generalisation:

[J]ust as social interaction is impossible without signs, it is also impossible without meaning. To communicate an experience of some other content of consciousness to another person, it must be related to a class or group of phenomena... this requires generalization. Social interaction presupposes generalization and the development of verbal meaning; generalization becomes possible only with the development of social interaction (1987, p.48).

Within connectionist teaching there is an encouragement of mathematical dialogue, partly to promote an active role for students counter to the monologic drill and copy of transmissionism, but also due to this key link with the process of generalisation, particularly through reasoning justification and argument within modelling and problem solving.

There is a discussion within ‘Thinking and speech’ on the contrast between oral and written speech (p. 203), with writing seen as requiring a higher level of abstraction due to its lacking of intonation and expression, and the absence of an interlocutor with their shared contextual knowledge. Also, the more immediate motivational factors present in speech are seen to be absent:

With every moment, the situation that is inherent in oral speech creates the motivation for each turn of speech; it creates the motivation for each segment of conversation or dialogue. The need for something produces the request. The question creates the answer. The expression brings the retort and the failure to understand the clarification. A multitude of similar relationships between speech and motive are fully determined by the situation inherent in real oral speech. Thus, oral speech is regulated by the dynamics of the situation. It flows entirely from the situation in accordance with this type of situational-motivational and situational-conditioning process. With written speech, on the other hand, we are forced to create the situation or – more accurately – to represent it in thought.
This distinction seems a useful one, and the developmental role of this generalisation of generalisations would apply to written mathematics also. However, the making of the distinction perhaps underestimates the potential variations within different forms of oral speech itself. In reasoning and justification it is not only the more immediate contextual factors which are present but also, these are present in relation to elements of mathematical systems. This can make these oral forms particularly conducive to generalisation and conceptual development, playing a similar role to writing in encouraging representation in thought at a distance from immediate contextual factors.

2.5 Conscious awareness and control

Intimately connected to dialogue and reasoning within connectionist teaching is an emphasis on reflexivity. Discussions of the appropriateness of strategies and the act of explaining and comparing methods encourage students' conscious awareness and greater control of their mathematical activity, as well as of their learning in a wider sense. This reflexivity, or metacognition, is in turn intimately connected with generalisation and the systemic nature of concepts.

In Vygotsky’s terms, conscious awareness represents the generalisation or abstraction of internal mental forms of activity (1987, p.190) and therefore systematisation:

[I]f a higher concept arises above the given concept, there must be several subordinate concepts that include it. Moreover, the relationships of these other subordinate concepts to the given concept must be defined by the system created by the higher concept. If this were not so, the higher concept would not be higher than the given concept. This higher concept presupposes both a hierarchical system and concepts subordinate and systematically related to the given concept. Thus, the generalization of the concept leads to its localization within a definite system of relationships of generality. These relationships are the foundation and the most natural and important connections among concepts. Thus, at one and the same time, generalization implies the conscious awareness and the systematization of concepts (1987, p. 191).

This means that,

[O]nly within a system can the concept acquire conscious awareness and a voluntary nature. Conscious awareness and the presence of a system are synonyms when we are speaking of concepts, just as spontaneity, lack of conscious awareness, and the absence of a system are three different words for designating the nature of the child’s concept (1987, p. 191).

This encourages the practice of paying some attention to the systemic connections of mathematics within pedagogy. The generalisation, for example, of proportional thinking from the various topics mentioned earlier leads to greater awareness, freedom of operation and control in thinking about those particular topics, in a similar way to how the learning of a foreign language as an adult can raise the consciousness and use of grammar in a first language through the generalisations which are made (see p. 180).

Mathematical reasoning, contrasting of methods and also exploration of misconceptions are also part of this relationship between reflexivity and systematisation: For example:
For contradiction to be sensed, the two contradictory judgments must be viewed as particular cases of a single, more general concept. As we have seen, this type of relationship among concepts is absent where concepts are not included in some system. It is, indeed, impossible (1987, p. 235).

In many ways it is difficult to separate mathematical metacognition, in the sense of hierarchical systemic concepts, from the metacognition of doing mathematics. As Veenman et al. (2006, p. 5) point out, 'It is very hard to have adequate metacognitive knowledge of one's competencies in a domain without substantial (cognitive) domain-specific knowledge'. The metacognitive aspects of learning mathematics are also intricately connected with these for a similar reason. These connections in the forms of metacognition are underlined by Vygotsky's argument that awareness and control, in a general sense, are brought in, developmentally, through the introduction of scientific concepts primarily in educational settings. However, it is also worth considering what the relevant system is in the case of learning mathematics, and how much conscious awareness and control of learning can arise if the system is relatively undeveloped.

Arguably this entire paper represents an attempt to provide just such a systematisation of learning, or at least the beginnings of a systematisation of some central aspects of the process. A systematisation of this nature, in some form, could then potentially aid teachers in developing more conscious control of practice. Although this systematisation shouldn't be inflicted on students, or even teachers for that matter, in an academic form such as this, discussions on what mathematics is, on what modelling involves, on how mathematics relates to the real world, on comparing strategies in problem solving and on what counts as a good mathematical justification etcetera are certainly possible, and some attention to those questions could help make school mathematics a more meaningful activity.

2.6 The relationship between everyday and scientific concepts

The final connections to discuss which appear in pedagogy are the connections between mathematics and students' own knowledge. This has a strong overlap with the section on connections to the real world and the motivations which arise from relating mathematics to relevant aspects of students' lives. It also encompasses students' previous mathematical conceptualisations, which can either be built upon or challenged. And, finally, it includes relationships with the generalisations which students have made in their everyday lives, either through their own activity or in relation to wider issues.

A starting point for exploring these connections is Vygotsky's distinction between everyday, or spontaneous, concepts. The key difference between the two forms of concept being the presence or absence of a system.

Outside a system, the only possible connections between concepts are those that exist between the objects themselves, that is, empirical connections. This is the source of the dominance of the logic of action and of syncretic connections of impressions in early childhood. Within a system, relationships between concepts begin to emerge. These relationships mediate the concept's relationship to the object through its relationship to other concepts. A different relationship between
the concept and the object develops. Supra – empirical connections between concepts become possible (1987, p.234).

Everyday concepts generally arise in the immediate presence of the aspect of reality they represent, even if, for the child, they are introduced by adults. Scientific concepts on the other hand are initially introduced, particularly in schooling, mediated by other concepts and 'the child’s thought is presented with different tasks than when his thought is left to itself’ (1987, p.178). This different relationship to the object of the two types of concept means that the developmental paths they follow are different.

When the child learns a scientific concept, he quickly begins to master the operations that are the fundamental weakness of the everyday concept. He easily defines the concept, applies it in various logical operations, and identifies its relationships to other concepts. We find the weakness of the scientific concept where we find the strength of the everyday concept, that is, in its spontaneous usage, in its application to various concrete situations, in the relative richness of its empirical content, and in its connections with personal experience. Analysis of the child’s spontaneous concept indicates that he has more conscious awareness of the object than of the concept itself. Analysis of his scientific concept indicates that he has more conscious awareness of the concept than of the object that is represented by it (1987, p. 218).

However, in development, both types of concept are part of a unified process, and are not separated in consciousness. The introduction of scientific concepts is dependant on the sufficient development of the everyday concepts which mediate their relationship to the object. Then, in the other direction, the 'system that emerges in the sphere of scientific concepts – is transferred structurally to the domain of everyday concepts, restructuring the everyday concept and changing its internal nature from above' (1987, p. 192). This two-way process is essential to Vygotsky’s argument that schooling (and therefore the social) leads development: 'conscious awareness enters through the gate opened up by the scientific concept' (p. 191). But it also stresses that the sometimes initial verbalism of scientific concepts can be made more concrete through their use in activity.

In some ways this reads like a happy ending. For Vygotsky, by a certain point in schooling the two lines of thought have merged and the distinction between the two types of concept seems redundant. However, he also makes a point that seems to contradict this. That is, that 'more elementary forms continue to predominate in many domains of experience for a long time' (p. 160), and, 'when applied in the domain of life experience, even the concepts of the adult and adolescent frequently fail to rise higher than the level of the pseudo-concept. This indicates that there is scope for the bi-directional process described in development to be an ongoing one in education (and life). It also raises the question though of why such forms of thinking generally dominate. The answer to this is perhaps essentially contained in the preceding argument. The situated cognition literature has been noted already and comparisons made with the memory-based empty formalism of the transmissionist teaching of mathematics. Schooling, and often everyday life, puts tasks before the individual that are not, in general, of a nature to require much development or use of scientific generalisations and systematisation in Vygotsky’s sense. In much of day-to-day life this is not necessarily a negative thing, but it is in schools, which could and should be spaces for developing rich experience of this type of active and developed thought.
2.6 Conclusion

The aim of this paper has been to show, first, that the various elements of Vygotsky’s theory form an interrelated whole. His analysis of the origin and development of thinking in concepts leads to a view of scientific concepts as primarily representing generalisation through abstraction, where the abstract is not empty formalism but contains within it both the rich concrete which has been generalised and systemic relationships with other concepts; that such generalisations only arise when there is a real need in problem solving, justifying or communicating in social activity; and that this generalisation represents conscious awareness and allows conscious control over our thought and activity.

Secondly, this interrelated theory has been shown to map across to the key elements of connectionist pedagogy, helping to begin to explain why and how these elements work, or why learning may be less effective in their absence. In doing so, an interrelated pedagogical system has been presented. This is not intended to be prescriptive. There are many ways to bring genuine problems, aspects of the real world, dialogue and attention to systemic connections into teaching. However, this is an argument for ensuring all those elements are present and to see the ‘systems’, that is, of mathematics, of doing mathematics and of learning mathematics, as interrelated and mutually beneficial.

The paper has used Vygotsky’s writings as a basis, and in small ways extended his analysis, in order to begin to structure a systematised understanding of the learning of mathematics. In the process much has been ignored. First, in the presentation of Vygotsky’s ideas, the basis in experiments and the theoretical justifications in his work have, in the main, been sacrificed in order to present the structural essence of his argument. In discussing pedagogy also, some statements have been made rather boldly about what is good and what is not. The justification for this approach is first, in maintaining an uncluttered focus throughout on the interrelation between theory and practice, and second, in allowing the system and its connections, as built through the paper, to take a sizeable share of the sense making burden.

By way of a conclusion and, in a spirit of over-reaching, it is worth noting that these two justifications look remarkably similar to the two central strands of Vygotsky’s theory: the interrelation of abstract and concrete and the role of systems. It has been noted already that some of the aspects of the theory were prefigured in Vygotsky’s grappling with the possibilities of developing a general science of psychology. If there is an aim to develop a science of the teaching and learning of mathematics then his ideas may equally provide a useful starting point.
References


